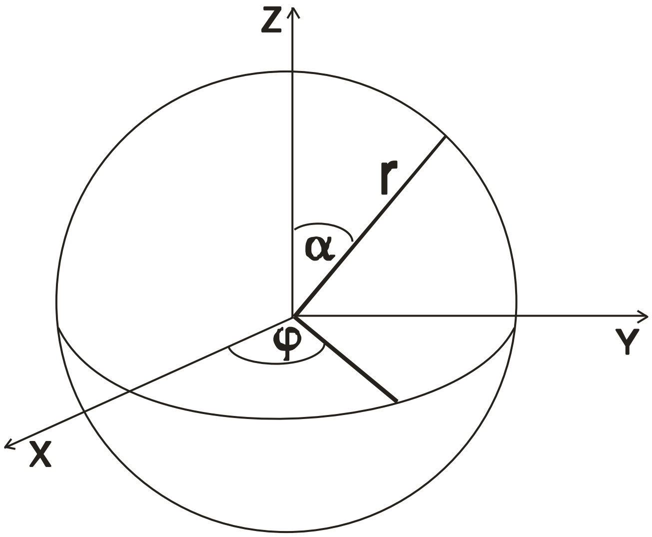
Coordenadas esféricas

Elemento de área de una esfera:



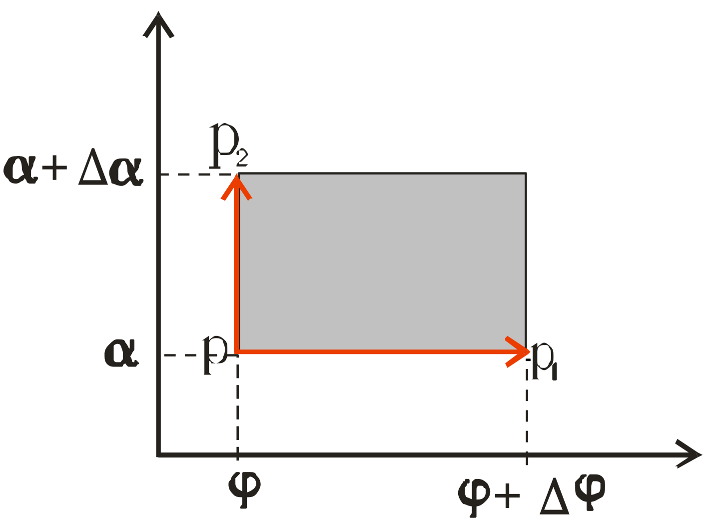
(r,φ,α)

$\hat{i} = rsen\alpha\cos\varphi$

$\hat{j} = rsen\alpha sen\varphi$

$\hat{k} = r\cos\alpha$

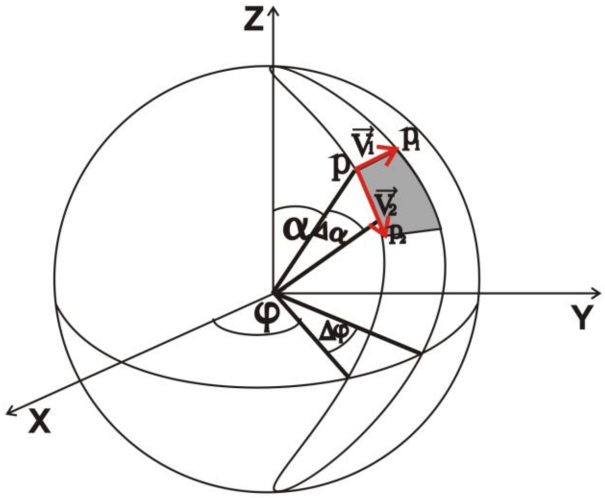
Si r se mantiene constante:



$P\quad \varphi,\alpha$

$P_1\quad \varphi+\Delta\varphi,\alpha$

$P_2\quad \varphi,\alpha+\Delta\alpha$



$$R\ \ \varphi,\alpha\ \ = p = \ \ rsen\alpha\cos\varphi, rsen\alpha sen\varphi, r\cos\alpha$$

$$R_1\ \ \varphi+\Delta\varphi,\alpha\ \ = p_1 = \ \ rsen\alpha\cos\ \ \varphi+\Delta\varphi\ \ , rsen\alpha sen\ \ \varphi+\Delta\varphi\ \ , r\cos\alpha$$

$$R_2\ \ \varphi,\alpha+\Delta\alpha\ \ = p_2 = \ \ rsen\ \ \alpha+\Delta\alpha\ \ \cos\varphi, rsen\ \ \alpha+\Delta\alpha\ \ sen\varphi, r\cos\ \ \alpha+\Delta\alpha$$

$$\vec{V}_1 = p_1 - p = R_1\ \ \varphi+\Delta\varphi,\alpha\ \ - R\ \ \varphi,\alpha\ \ = \frac{\partial R}{\partial\varphi}\Delta\varphi \qquad \vec{V}_2 = p_2 - p = R_1\ \ \varphi,\alpha+\Delta\alpha\ \ - R\ \ \varphi,\alpha\ \ = \frac{\partial R}{\partial\alpha}\Delta\alpha$$

$$\lim_{\varphi\to 0}\frac{\partial R}{\partial\varphi}\Delta\varphi = \frac{\partial R}{\partial\varphi}d\varphi \qquad \lim_{\alpha\to 0}\frac{\partial R}{\partial\alpha}\Delta\alpha = \frac{\partial R}{\partial\alpha}d\alpha$$

$$d\vec{A}_1 = \vec{V}_1\times\vec{V}_2 = \frac{\partial R}{\partial\varphi}d\varphi\times\frac{\partial R}{\partial\alpha}d\alpha = \left(\frac{\partial R}{\partial\varphi}\times\frac{\partial R}{\partial\alpha}\right)d\varphi d\alpha$$

$$\frac{\partial R}{\partial\varphi}\times\frac{\partial R}{\partial\alpha}\to\begin{bmatrix} -rsen\alpha sen\varphi & rsen\alpha\cos\varphi & 0 \\ r\cos\alpha\cos\varphi & r\cos\alpha sen\varphi & -rsen\alpha \end{bmatrix}$$

$$\hat{i} = -r^2 sen^2\alpha\cos\varphi$$

$$\hat{j} = -r^2 sen^2\alpha sen\varphi$$

$$\hat{k} = \ \ -r^2 sen^2\varphi sen\alpha\cos\alpha - r^2\cos^2\varphi sen\alpha\cos\alpha$$

$$= \ \ -r^2 sen\alpha\cos\alpha\ \ sen^2\varphi + \cos^2\varphi$$

$$= \ \ -r^2 sen\alpha\cos\alpha$$

$$\sqrt{\hat{i}^2+\hat{j}^2+\hat{k}^2}=\sqrt{r^4sen^4\alpha\cos^2\varphi+r^4sen^4\alpha sen^2\varphi+r^4sen^2\alpha\cos^2\alpha}$$

$$=\sqrt{r^4sen^4\alpha\;\;\cos^2\varphi+sen^2\varphi\;\;+r^4sen^2\alpha\cos^2\alpha}$$

$$=\sqrt{r^4sen^4\alpha+r^4sen^2\alpha\cos^2\alpha}$$

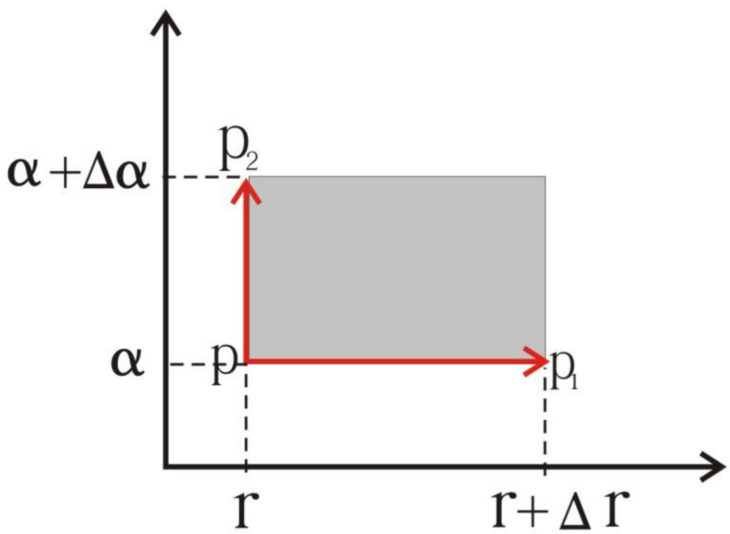
$$=\sqrt{r^4sen^2\alpha\;\;sen^2\alpha+\cos^2\alpha}$$

$$=\sqrt{r^4sen^2\alpha}$$

$$=r^2sen\alpha$$

- $d\overrightarrow{A_1}=\;r^2sen\alpha\;\;d\varphi d\alpha$

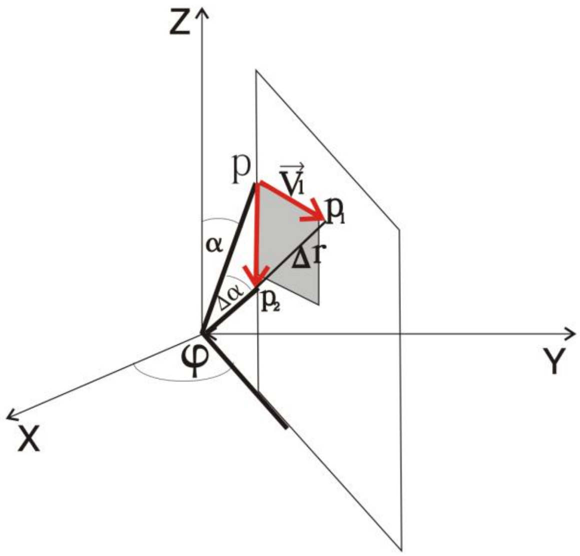
Si δ se mantiene constante:



$P\;\;r,\alpha$

$P_1\;\;r+\Delta r,\alpha$

$P_2\;\;r,\alpha+\Delta\alpha$



$$R\;\; r,\alpha \;\; = p = \;\; rsen\alpha\cos\varphi, rsen\alpha sen\varphi, r\cos\alpha$$

$$R_1\;\; r+\Delta r,\alpha \;\; = p_1 = \;\;\; r+\Delta r \;\; sen\alpha\cos\varphi, \;\; r+\Delta r \;\; sen\alpha sen\varphi, \;\; r+\Delta r \;\; \cos\alpha$$

$$R_2\;\; r,\alpha+\Delta\alpha \;\; = p_2 = \;\; rsen \;\; \alpha+\Delta\alpha \;\; \cos\varphi, rsen \;\; \alpha+\Delta\alpha \;\; sen\varphi, r\cos \;\; \alpha+\Delta\alpha$$

$$\vec{V_1} = p_1 - p = R_1 \;\; r+\Delta r,\alpha \;\; - R \;\; r,\alpha \;\; = \frac{\partial R}{\partial r}\Delta r \qquad \vec{V_2} = p_2 - p = R_1 \;\; \varphi,\alpha+\Delta\alpha \;\; - R \;\; \varphi,\alpha \;\; = \frac{\partial R}{\partial \alpha}\Delta\alpha$$

$$\lim_{r\to 0}\frac{\partial R}{\partial r}\Delta r = \frac{\partial R}{\partial r}dr \qquad \lim_{\alpha\to 0}\frac{\partial R}{\partial \alpha}\Delta\alpha = \frac{\partial R}{\partial \alpha}d\alpha$$

$$d\overrightarrow{A_2} = \vec{V}_1 \times \vec{V}_2 = \frac{\partial R}{\partial r}dr \times \frac{\partial R}{\partial \alpha}d\alpha = \left(\frac{\partial R}{\partial r}\times\frac{\partial R}{\partial \alpha}\right)drd\alpha$$

$$\frac{\partial R}{\partial r}\times\frac{\partial R}{\partial \alpha} \to \begin{bmatrix} sen\alpha\cos\varphi & sen\alpha sen\varphi & \cos\alpha \\ r\cos\alpha\cos\varphi & r\cos\alpha sen\varphi & -rsen\alpha \end{bmatrix}$$

$$\hat{i} = -rsen^2\alpha sen\varphi - rsen\varphi\cos^2\alpha$$

$$= -rsen\varphi \;\; sen^2\alpha + \cos^2\alpha$$

$$= -rsen\varphi$$

$$\hat{j} = rsen^2\alpha\cos\varphi + r\cos^2\alpha\cos\varphi$$

$$= r\cos\varphi \;\; sen^2\alpha + \cos^2\alpha$$

$$= r\cos\varphi$$

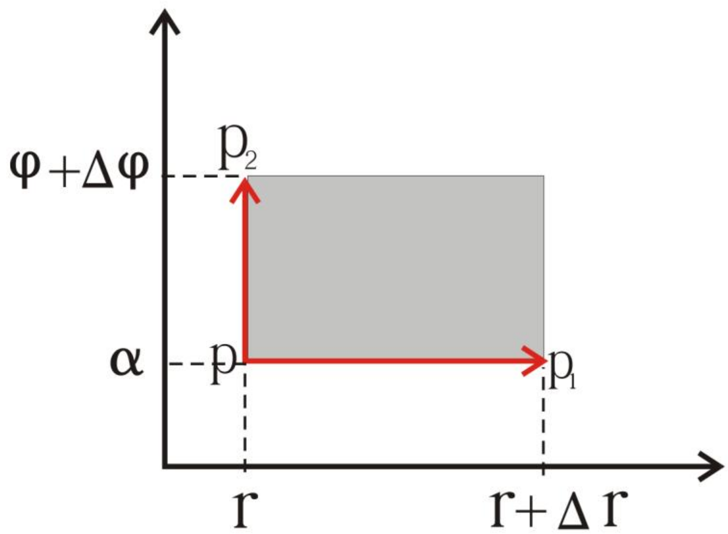
$$\hat{k} = r\cos\alpha\cos\varphi sen\alpha sen\varphi - r\cos\alpha\cos\varphi sen\alpha sen\varphi$$

$$= 0$$

$$\sqrt{\hat{i}^2+\hat{j}^2+\hat{k}^2}=\sqrt{r^2 sen^2\varphi+r^2\cos^2\varphi}$$
$$=\sqrt{r^2\ \ sen^2\varphi+\cos^2\varphi}$$
$$=\sqrt{r^2}$$
$$=r$$

- $d\overrightarrow{A_2}=\ \ r\ \ drd\alpha$

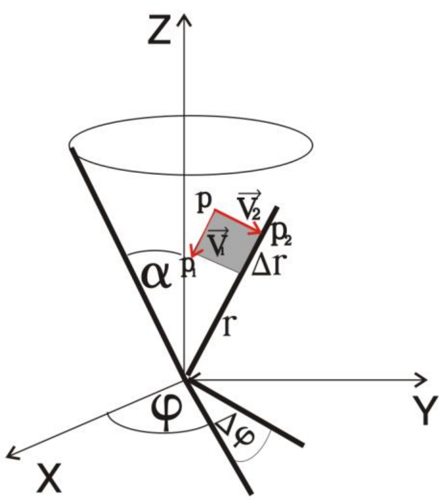
Si se α mantiene constante:



$P\ \ r,\varphi$

$P_1\ \ r+\Delta r,\varphi$

$P_2\ \ r,\varphi+\Delta\varphi$



$R\ \ r,\varphi\ \ =p=\ \ rsen\alpha\cos\varphi,rsen\alpha sen\varphi,r\cos\alpha$

$R_1\ \ r+\Delta r,\varphi\ \ =p_1=\ \ \ r+\Delta r\ \ sen\alpha\cos\varphi,\ \ r+\Delta r\ \ sen\alpha sen\varphi,\ \ r+\Delta r\ \ \cos\alpha$

$R_2\ \ r,\varphi+\Delta\varphi\ \ =p_2=\ \ rsen\alpha\,\mathrm{co}\ \ \mathrm{s}\,\varphi+\Delta\varphi\ \ ,rsen\alpha sen\ \ \varphi+\Delta\varphi\ \ ,r\cos\alpha$

$$\vec{V_1} = p_1 - p = R_1(r+\Delta r,\varphi) - R(r,\varphi) = \frac{\partial R}{\partial r}\Delta r$$

$$\vec{V_2} = p_2 - p = R_2(r,\varphi+\Delta\varphi) - R(\varphi,\alpha) = \frac{\partial R}{\partial \varphi}\Delta\varphi$$

$$\lim_{r\to 0}\frac{\partial R}{\partial r}\Delta r = \frac{\partial R}{\partial r}dr$$

$$\lim_{\varphi\to 0}\frac{\partial R}{\partial \varphi}\Delta\varphi = \frac{\partial R}{\partial \varphi}d\varphi$$

$$d\vec{A_3} = \vec{V}_1\times\vec{V}_2 = \frac{\partial R}{\partial r}dr\times\frac{\partial R}{\partial \varphi}d\varphi = \left(\frac{\partial R}{\partial r}\times\frac{\partial R}{\partial \varphi}\right)drd\varphi$$

$$\frac{\partial R}{\partial r}\times\frac{\partial R}{\partial \varphi}\to\begin{bmatrix} sen\alpha\cos\varphi & sen\alpha sen\varphi & \cos\alpha \\ -rsen\alpha sen\varphi & rsen\alpha\cos\varphi & 0\end{bmatrix}$$

$$\hat{i} = -rsen\alpha\cos\alpha\cos\varphi$$

$$\hat{j} = -rsen\alpha\cos\alpha sen\varphi$$

$$\hat{k} = rsen^2\alpha\cos^2\varphi + rsen^2\alpha sen^2\varphi$$

$$= rsen^2\alpha(\cos^2\varphi + sen^2\varphi)$$

$$= rsen^2\alpha$$

$$\sqrt{\hat{i}^2+\hat{j}^2+\hat{k}^2} = \sqrt{r^2sen^2\alpha\cos^2\alpha\cos^2\varphi + r^2sen^2\alpha\cos^2\alpha sen^2\varphi + r^2sen^4\alpha}$$

$$= \sqrt{r^2sen^2\alpha\cos^2\alpha(sen^2\varphi+\cos^2\varphi) + r^2sen^4\alpha}$$

$$= \sqrt{r^2sen^2\alpha\cos^2\alpha + r^2sen^4\alpha}$$

$$= \sqrt{r^2sen^2\alpha(\cos^2\alpha + sen^2\alpha)}$$

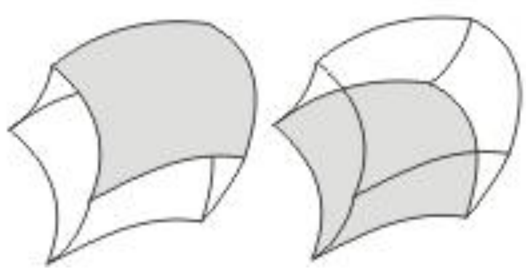
$$= \sqrt{r^2sen^2\alpha}$$

$$= rsen\alpha$$

- $d\vec{A_3} = (rsen\alpha)\,drd\varphi$

Flujo en elemento de volumen de una esfera

Flujo 1:



$R(r+\Delta r,\varphi,\alpha)\vec{A}_1$ $\quad$ $R(r,\varphi,\alpha)\vec{A}_1$

$$F_1 = R(r+\Delta r,\varphi,\alpha)\vec{A}_1 - R(r,\varphi,\alpha)\vec{A}_1$$

$$= \left[R(r+\Delta r,\varphi,\alpha) - R(r,\varphi,\alpha)\right]\vec{A}_1$$

$$= \left(\frac{\partial R}{\partial r}dr\right)\vec{A}_1$$

$$= \left(\frac{\partial R}{\partial r}dr\right)r^2sen\alpha d\alpha d\varphi$$

$$= \left(\frac{\partial R}{\partial r}r^2sen\alpha\right)drd\alpha d\varphi$$

Flujo 2:

$R(r,\varphi+\Delta\varphi,\alpha)\vec{A}_2$

$R(r,\varphi,\alpha)\vec{A}_2$

$$F_2 = R(r,\varphi+\Delta\varphi,\alpha)\vec{A}_2 - R(r,\varphi,\alpha)\vec{A}_2$$
$$= \left[R(r,\varphi+\Delta\varphi,\alpha) - R(r,\varphi,\alpha)\right]\vec{A}_2$$
$$= \left(\frac{\partial R}{\partial \varphi}d\varphi\right)\vec{A}_2$$
$$= \left(\frac{\partial R}{\partial \varphi}d\varphi\right)rd\alpha dr$$
$$= \left(\frac{\partial R}{\partial \varphi}r\right)drd\alpha d\varphi$$

Flujo 3:

$R(r,\varphi,\alpha+\Delta\alpha)\vec{A}_3$

$R(r,\varphi,\alpha)\vec{A}_3$

$$F_3 = R(r,\varphi,\alpha+\Delta\alpha)\vec{A}_3 - R(r,\varphi,\alpha)\vec{A}_3$$
$$= \left[R(r,\varphi,\alpha+\Delta\alpha) - R(r,\varphi,\alpha)\right]\vec{A}_3$$
$$= \left(\frac{\partial R}{\partial \alpha}d\alpha\right)\vec{A}_3$$
$$= \left(\frac{\partial R}{\partial \alpha}d\alpha\right)rsen\alpha drd\varphi$$
$$= \left(\frac{\partial R}{\partial \alpha}rsen\alpha\right)d\alpha drd\varphi$$

En resumen:

$$F_{neto} = F_1 + F_2 + F_3$$
$$= \left(\frac{\partial R}{\partial r}r^2 sen\alpha\right)drd\alpha d\varphi + \left(\frac{\partial R}{\partial \varphi}r\right)drd\alpha d\varphi + \left(\frac{\partial R}{\partial \alpha}rsen\alpha\right)d\alpha drd\varphi$$
$$= \left(\frac{\partial R}{\partial r}r^2 sen\alpha + \frac{\partial R}{\partial \varphi}r + \frac{\partial R}{\partial \alpha}rsen\alpha\right)drd\alpha d\varphi$$

$$\iiint div(\vec{F})dv = \iint Fd\vec{A} \Rightarrow div(\vec{F})dv = Fd\vec{A}$$

$$div(\vec{F}) = \frac{Fd\vec{A}}{dv}$$

Considerando que el volumen es:

$$dv=\left(\frac{\partial R}{\partial r}dr\times\frac{\partial R}{\partial \alpha}d\alpha\right)\bullet\frac{\partial R}{\partial \varphi}d\varphi$$

$$=\left[\left(\frac{\partial R}{\partial r}\times\frac{\partial R}{\partial \alpha}\right)\bullet\frac{\partial R}{\partial \varphi}\right]drd\alpha d\varphi$$

$$=\left[\ -rsen\varphi, r\cos\varphi, 0\ \bullet\ -rsen\alpha sen\varphi, rsen\alpha\cos\varphi, 0\ \right]drd\alpha d\varphi$$

$$=\left[r^2 sen\alpha sen^2\varphi+r^2\cos^2\varphi sen\alpha\right]drd\alpha d\varphi$$

$$=\left[r^2 sen\alpha\ \ sen^2\varphi+\cos^2\varphi\ \right]drd\alpha d\varphi$$

$$=\ r^2 sen\alpha\ \ drd\alpha d\varphi$$

Finalmente, la divergencia será:

$$div\ \vec{F}\ =\frac{Fd\vec{A}}{dv}=\frac{\left(\frac{\partial R}{\partial r}r^2 sen\alpha+\frac{\partial R}{\partial \varphi}d\varphi+\frac{\partial R}{\partial \alpha}rsen\alpha\right)drd\alpha d\varphi}{r^2 sen\alpha\ \ drd\alpha d\varphi}$$

$$=\frac{\frac{\partial Rr^2 sen\alpha}{\partial r}+\frac{\partial Rr}{\partial \varphi}+\frac{\partial Rrsen\alpha}{\partial \alpha}}{r^2 sen\alpha}$$

$$=\frac{\frac{\partial Rr^2 sen\alpha}{\partial r}}{r^2 sen\alpha}+\frac{\frac{\partial Rr}{\partial \varphi}}{r^2 sen\alpha}+\frac{\frac{\partial Rrsen\alpha}{\partial \alpha}}{r^2 sen\alpha}$$

$$=\frac{1}{r^2}\frac{\partial Rr^2}{\partial r}+\frac{1}{rsen\alpha}\frac{\partial R}{\partial \varphi}+\frac{1}{rsen\alpha}\frac{\partial Rsen\alpha}{\partial \alpha}$$